10/4/21

↓ use first point

Ex: Estimate Δf from $(1,1,6)$ $(1.5, 1.5, 5.5)$

Sol: $\Delta f \approx df$ where $dx_i \approx \Delta x_i$

so $\Delta f \approx f_x(1,1,6)\Delta x + f_y(1,1,6)\Delta y + f_z(1,1,6)\Delta z$

$= e(1.5-1) + 2e(1.5-1) + \frac{1}{2}e(5.5-6)$

$= \frac{1}{2}e + \frac{2}{2}e - \frac{1}{4}e = \frac{5}{4}e$

10/6/21 Multivariate Chain Rule:

goal: extend the chain rule from calculus 1 to multivariate functions

Composition of multivariate function

(has "n" variables)

give a function $f: D \in \mathbb{R}^n \to \mathbb{R}$

so $f(x_1, x_2, \ldots, x_n)$

To generalize composition of calculus 1, we will allow each coordinate x_i to be a function of other variables

ex: $x_i = g_i(t_1, t_2, \ldots, t_k)$

Ex: Let $f(x,y,z) = xy + yz - z^2$

and $x(s,t) = s-t$

$y(s,t) = s^2+t$

$z(s,t) = \cos(t)$

The composition $f(x(s,t), y(s,t), z(s,t))$ has formula:

$f(s-t, s^2+t, \cos(t))$

$= (s-t)(s^2+t) + (s^2+t)(\cos(t)) - \cos^2(t)$ (could simplify)

Observation: If $f: \mathbb{R}^n \to \mathbb{R}$ and $g_i: \mathbb{R}^k \to \mathbb{R}$ for $1 \le i \le n$, the composition $f(g_1(s_1, s_2, \ldots s_k), g_2(s_1, s_2 \ldots s_k) \ldots g_n(s_1, s_2, \ldots s_k))$ is a function of k variables

$\mathbb{R}^k \xrightarrow{g_i} \mathbb{R} \to \mathbb{R}^n \xrightarrow{f} \mathbb{R}$ $\quad$ $\mathbb{R}^k \xrightarrow{g_i} \mathbb{R}^n \xrightarrow{f} \mathbb{R}$

10/6/21

Q: How do we understand derivatives?

Definition: a function $f: D \subseteq \mathbb{R}^n \to \mathbb{R}$ is differentiable at point $p \in D$ (domain) when f is "well approximated" by its tangent (hyper)plane at p

Note: This notion is (basically) the same notion from Calc 1

Suppose $f(x,y)$ and $x(t)$, $y(t)$ are differentiable. Then, near a point $p=(a,b)$, $f(x,y) = f(a,b) + (f_x(a,b) + \varepsilon_x)(x-a) + (f_y(a,b) + \varepsilon_y)(y-b)$

error in x / error in y

where $(\varepsilon_x, \varepsilon_y) \to (0,0)$ as $(x,y) \to (a,b)$

Let t_0 be a time so that $(x(t_0), y(t_0)) = (a,b)$
our tangent plane (evaluated along $(x(t), y(t))$) becomes:

$$f(x(t), y(t)) = f(x(t_0), y(t_0)) + (f_x(x(t_0), y(t_0)) + \varepsilon_x)(x(t) - x(t_0)) + (f_y(x(t_0), y(t_0)) + \varepsilon_y)(y(t) - y(t_0))$$

$$f(x(t), y(t)) - f(x(t_0), y(t_0)) = f_x(x(t_0), y(t_0))(x(t) - x(t_0)) + f_y(x(t_0), y(t_0))(y(t) - y(t_0)) + \varepsilon_x^{(t)}(x(t) - x(t_0)) + \varepsilon_y^{(t)}(y(t) - y(t_0))$$

Dividing both sides by $t - t_0$ (when $t \neq t_0$):

$$\frac{f(x(t), y(t)) - f(x(t_0), y(t_0))}{t - t_0} = \underbrace{f_x(x(t_0), y(t_0))}_{\text{constant}}\left(\frac{x(t) - x(t_0)}{t - t_0}\right) + \underbrace{f_y(x(t_0), y(t_0))}_{\text{constant}}\left(\frac{y(t) - y(t_0)}{t - t_0}\right) + \varepsilon_x^{(t)}\left(\frac{x(t) - x(t_0)}{t - t_0}\right) + \varepsilon_y^{(t)}\left(\frac{y(t) - y(t_0)}{t - t_0}\right)$$

$$= \lim_{t \to t_0} \varepsilon_y^{(t)} \cdot \lim_{t \to t_0} \left(\frac{y(t) - y(t_0)}{t - t_0}\right)$$

limiting $t \to t_0$ we obtain

$$\frac{\partial}{\partial t}\left[f(x(t), y(t))\right]\Big|_{t=t_0} = \lim_{t \to t_0} \text{ of } \ldots$$

$$= f_x(x(t_0), y(t_0))\, x'(t_0) + f_y(x(t_0), y(t_0))\, y'(t_0) + \lim_{t \to t_0} \varepsilon_x(t_0)\, x'(t) + \lim_{t \to t_0} \varepsilon_y(t)\, y'(t_0)$$

$\lim_{t \to t_0}$

10/6/21

hence $\frac{\partial}{\partial t}\left[f(x(t), y(t))\right]\Big|_{t=t_0} = f_x(x(t_0), y(t_0))x'(t_0) + f_y(x(t_0), y(t_0))y'(t_0)$

The derivation just performed can be generalized to prove:

prop (multivariate chain rule): Suppose $f(x_1, x_2 \dots x_n)$ and $x_i = x_i(t_1, t_2, \dots t_m)$, are differentiable. Then:

$$\frac{\partial f}{\partial t_j} = \frac{\partial f}{x_1}\cdot\frac{\partial x_1}{\partial t_j} + \frac{\partial f}{x_2}\cdot\frac{\partial x_2}{\partial t_j} + \dots\dots \frac{\partial f}{\partial x_n}\cdot\frac{\partial x_n}{\partial t_j}$$

Comment: "crossing out" the ∂x_i's is not ok, b/c then the formula is meaningless!

Ex: compute $\frac{\partial f}{r}, \frac{\partial f}{s}, \frac{\partial f}{t}$ for $f(x,y,z) = x^4y + y^2z^3$, $x(r,s,t) = rse^t$, $y(r,s,t) = rs^2e^{-t}$, $z(r,s,t) = r^2s\sin(t)$

Sol 1 - w/o chain rule

$f(x,y,z) = f(rse^t, rs^2e^{-t}, r^2s\sin(t))$

$= (rse^t)^4(rs^2e^{-t}) + (rs^2e^{-t})^2(r^2s\sin(t))^3$

$= r^5s^6e^{3t} + r^8s^7e^{-2t}\sin^3(t)$

$\frac{\partial f}{\partial r} = 5r^4s^6e^{3t} + 8r^7s^7e^{-2t}\sin^3(t)$

$\frac{\partial f}{\partial s} = 6r^5s^5e^{3t} + 7r^8s^6e^{-2t}\sin^3(t)$

$\frac{\partial f}{\partial t} = 3r^5s^6e^{3t} + r^8s^7\left(-2e^{-2t}\sin^3(t) + e^{-2t}3\sin^2(t)\cos(t)\right)$

Sol 2 (w/ chain rule): by chain rule: $\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial r}$

$\frac{\partial f}{\partial x} = 4x^3y = 4(rse^t)^3(rs^2e^{-t}) = 4r^4s^5e^{2t}$

$\frac{\partial f}{\partial y} = x^4 + 2yz^3 = (rse^t)^4 + 2(rs^2e^{-t})(r^2s\sin(t))^3 = r^4s^4e^{4t} + 2r^7s^5e^{-t}\sin^3(t)$

$\frac{\partial f}{\partial z} = 3y^2z^2 = 3(rs^2e^{-t})^2(r^2s\sin(t))^2 = 3r^6s^6e^{-2t}\sin^2(t)$

10/6/21

$\frac{\partial x}{\partial r} = se^{t}$ $\frac{\partial y}{\partial r} = s^2 e^{-t}$ $\frac{\partial z}{\partial r} = 2rs\sin(t)$

by chain rule

$\frac{\partial f}{\partial r} = (4r^3 s^5 e^{2t})(se^{t}) + (r^4 s^4 e^{4t} + 2r^7 s^5 e^{-t}\sin^3(t))(s^2 e^{-t})$
$+ (3r^6 s^6 e^{-2t}\sin^2(t))(2rs\sin(t))$

$= 5r^4 s^6 e^{3t} + 8r^7 s^7 e^{-2t}\sin^3(t)$

$\frac{\partial f}{\partial s}$ $\frac{\partial x}{\partial s} = re^{t}$, $\frac{\partial y}{\partial s} = 2rse^{-t}$, $\frac{\partial z}{\partial s} = r^2\sin(t)$

$\frac{\partial f}{\partial s} = (4r^4 s^5 e^{2t})(re^{t}) + (r^4 s^4 e^{4t} + 2r^7 s^5 e^{-t}\sin^3(t))(2rse^{-t})$
$+ (3r^6 s^6 e^{-2t}\sin^2(t))(r^2\sin(t))$

compute $\frac{\partial f}{\partial t}$: $\frac{\partial x}{\partial t} = rse^{t}$ $\frac{\partial y}{\partial t} = -rs^2 e^{-t}$, $\frac{\partial z}{\partial t} = r^2 s\cos(t)$

$\frac{\partial f}{\partial t} = (4r^4 s^5 e^{2t})(rse^{t}) + (r^4 s^4 e^{4t} + 2r^7 s^5 e^{-t}\sin^3(t))(-rs^2 e^{-t})$
$+ (3r^6 s^6 e^{-2t}\sin^2(t))(r^2\cos(t))$

exercise: repeat w/ both solutions for
$f(x,y) = e^x\sin(y)$, $x = st^2$, $y = s^2 t$
find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ (use chain rule first)

Q: Given an implicit (hyper)surface, how do we compute the slope of the tangent at a given point?

A: use the Implicit Function theorem (IFT).

Prop (Implicit function theorem): Suppose $F(x_1, x_2, \dots x_n)$ is differentiable on a disk containing point $\vec{p}$. Further suppose that $F(\vec{p}) = 0$ and $\frac{\partial F}{x_i}$ are continuous and $\left.\frac{\partial F}{\partial x_n}\right|_{\vec{p}} \neq 0$

then, near $\vec{p}$, $x_n = f(x_1, x_2, \dots x_{n-1})$ and for all i,

$$\frac{\partial f}{x_i} = \frac{-\partial F}{x_i} \Big/ \frac{\partial F}{\partial x_n}$$